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NEW CONNECTION FORMULAE BETWEEN CHEBYSHEV AND LUCAS POLYNOMIALS: NEW EXPRESSIONS INVOLVING LUCAS NUMBERS VIA HYPERGEOMETRIC FUNCTIONS

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Abstract. This article deals with the solutions of the connection problems between Lucas polynomials and the first and second kinds of Chebyshev polynomials. We show that the connection coefficients involve hypergeometric functions of the type ${}_2F_1$ of certain arguments. Thanks to the new derived connection formulae, new formulae of the famous Lucas numbers and their derivatives sequences are presented. As another application of the derived formulae, some new definite integrals involving Lucas and Chebyshev polynomials are evaluated.

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1. Introduction

The use of hypergeometric functions is a very important tool in various branches of pure and applied mathematics. These types of functions appear in many important problems related to special functions. For example, several linearization and connection problems can be solved via hypergeometric functions (see for instance [1]).

The applications of the celebrated integer sequences such as Fibonacci and Lucas numbers are of great importance in modern science, see for instance [17, 18]. There are several studies interested in developing various relations concerned with Fibonacci and Lucas numbers, see for instance, [25, 16, 24, 14, 15, 27, 26, 4].

The first and second kinds of Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are of fundamental importance, see for example [12, 19, 3]. These two families of polynomials are ultraspherical polynomials. There are other two kinds of Chebyshev polynomials, namely, Chebyshev polynomials of third and fourth kinds $V_n(x)$ and $W_n(x)$. These two families are not ultraspherical polynomials. In fact, they are two nonsymmetric families of Jacobi polynomials. There are several theoretical and practical studies about these two families, see for example [10, 2, 11]. A comprehensive study on the four families of Chebyshev polynomials can be found in the interesting book of Manson and Handscomb [21].

The connection coefficients play an essential part in various problems in mathematics and in mathematical physics. Several studies are interested in investigating the connection problem between two sets of orthogonal polynomials, (see, for instance, [5, 8, 9, 13, 20, 22, 23]) .

To be more precise, if we have the two sets of polynomials $\{A_i(x)\}_{i\geq 0}$ and $\{B_j(x)\}_{j\geq 0}$, then to solve the connection problem between them, we have to find the coefficients $C_{i,j}$ such that

$$A_i(x) = \sum_{j=0}^{i} C_{i,j}\, B_j(x). \tag{1}$$

The analytical formulae for the connection coefficients $C_{i,j}$ in (1) have been found by many techniques if $\{A_i(x)\}_{i\geq 0}$ and $\{B_j(x)\}_{j\geq 0}$ are orthogonal, see for example [9]. The most of these analytical formulae are expressed in terms of terminating hypergeometric series of various types. For instance, a terminating hypergeometric series of the type ${}_3F_2(1)$ appears in the solution of the Jacobi-Jacobi connection problem (see, [9]). Up to now, and to the best of our knowledge, the solutions of connection problems between Lucas polynomials and various orthogonal polynomials are traceless in literature. This gives us a motivation for investigating such problems.

In this article, we are interested in solving the connection problem (1) for the the four cases correspond to the four choices for the polynomials $A_i(x)$ and $B_j(x)$:

(i) $A_i(x) = L_i(x), B_j(x) = T_j(x)$, (ii) $A_i(x) = L_i(x), B_j(x) = U_j(x)$,

(iii) $A_i(x) = T_i(x), B_j(x) = L_j(x)$, (iv) $A_i(x) = U_i(x), B_j(x) = L_j(x)$,

where $L_i(x)$ is the Lucas polynomial of degree i, and $T_j(x)$ and $U_j(x)$ are, respectively, the Chebyshev polynomials of the first and second kinds, each of degree j.

The paper is organized as follows. In Section 2, an overview on Chebyshev polynomials, Lucas polynomials and Lucas numbers is presented. Section 3 is interested in stating and proving two theorems, in which the connection formulae between Chebyshev polynomials of the first and second kinds and Lucas polynomials are given. The inversion formulae of those obtained in Section 3 are given in Section 4. In Section 5, three applications based on the results obtained in Sections 3 and 4 are presented. In the first application, some new expressions involving the celebrated Lucas and Fibonacci numbers are given. In the second, some identities of the derivatives sequences of Lucas numbers are given. The third application is devoted to evaluating some definite integrals involving certain products of Chebyshev and Lucas polynomials.

2. An overview on Chebyshev and Lucas polynomials

In this section, some properties of Lucas polynomials and their related numbers are presented. In addition, some properties of Chebyshev polynomials of the first and second kinds are given.

2.1. Some properties of $T_i(x)$ and $U_i(x)$, $i \geq 0$.

Chebyshev polynomials of the first and second kinds $T_i(x)$ and $U_i(x)$ are defined as (see, [21]):

$$T_i(x) = \cos(i\,\theta),$$

and

$$U_i(x) = \frac{\sin(i+1)\theta}{\sin\theta},$$

where $x = \cos\theta$. The polynomials $T_i(x)$ and $U_i(x)$ may be constructed , respectively, by means of the recurrence relations:

$$T_i(x) = 2x\,T_{i-1}(x) - T_{i-2}(x), \quad i = 2, 3, \ldots,$$

with the initial values:

$$T_0(x) = 1, \quad T_1(x) = x,$$

and

$$U_i(x) = 2x\, U_{i-1}(x) - U_{i-2}(x), \quad i = 2, 3, \dots,$$

with the initial values:

$$U_0(x) = 1, \quad U_1(x) = 2x.$$

The analytic forms of $T_i(x)$ and $U_i(x)$ are:

$$T_i(x) = \frac{i}{2} \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(-1)^j}{i-j} \binom{i-j}{j} (2\,x)^{i-2\,j},$$

and

$$U_i(x) = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^j \binom{i-j}{j} (2\,x)^{i-2\,j},$$

where the notation $\lfloor z \rfloor$ represents the largest integer less than or equal to z.
The following specific formulae are useful in the sequel.

$$T_i(1) = 1, \quad U_i(1) = i + 1, \tag{2}$$

$$D^q T_i(1) = \prod_{j=0}^{q-1} \frac{(i-j)(i+j)}{2\,j+1}, \; q \geq 1, \tag{3}$$

and

$$D^q U_i(1) = (i+1) \prod_{j=0}^{q-1} \frac{(i-j)(i+j+2)}{2\,j+3}, \; q \geq 1. \tag{4}$$

2.2. **Lucas polynomials and their related numbers.** The Lucas polynomials may be constructed by mans of the recurrence relation:

$$L_{m+2}(x) = x\, L_{m+1}(x) + L_m(x), \; m \geq 0,$$

with the initial values

$$L_0(x) = 2, \quad L_1(x) = x.$$

The polynomials $L_m(x)$ have the following explicit form:

$$L_m(x) = \frac{\left(x + \sqrt{x^2+4}\right)^m - \left(x - \sqrt{x^2+4}\right)^m}{2^m},$$

and also the following power series representation:

$$L_m(x) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m}{m-j} \binom{m-j}{j} x^{m-2j}, \quad m \geq 1.$$

The first few Lucas polynomials are:

$$L_1(x) = x, \quad L_2(x) = x^2 + 2,$$

$$L_3(x) = x^3 + 3\,x, \quad L_4(x) = x^4 + 4\,x^2 + 2.$$

The Lucas numbers are denoted by L_m, and they are defined as

$$L_m = L_m(1),$$

and the Binet's form for the mth Lucas number is

$$L_m = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m.$$

The qth derivative sequence of the Lucas numbers is denoted by $L_m^{(q)}$. Explicitly

$$L_m^{(q)} = D^q\, L_m(x)\big|_{x=1}.$$

The Lucas numbers L_m have the following factorization formulae using complex numbers:

$$L_m = \prod_{k=0}^{m-1} \left(1 - 2\,i \cos\left(\frac{\pi(2k+1)}{2m}\right)\right),$$

where $i^2 = -1$.
The two families of Lucas and Fibonacci numbers are linked with Chebyshev polynomials of the first and second kinds of certain complex variables (see [6, 28]).

$$T_n\left(\frac{i}{2}\right) = \frac{i^n}{2}\, L_n, \tag{5}$$

$$T_n(-2\,i) = \frac{(-i)^n}{2}\, L_{3\,n}, \tag{6}$$

$$U_n\left(\frac{i}{2}\right) = i^n\, F_{n+1}, \tag{7}$$

$$U_n(-2\,i) = \frac{(-i)^n}{2}\, F_{3(n+1)}. \tag{8}$$

It is worthy to mention here that there are many articles investigate Fibonacci numbers and their derivatives sequences, see [15, 7]. For more properties of Fibonacci polynomials and numbers, see for example, the important book of Koshy [18].

2.3. **Generalized hypergeometric functions.** The generalized hypergeometric functions appear in many important problems related to special functions. The well-known definition of the generalized hypergeometric function is

$${}_pF_q\left(\begin{matrix} a_1, a_2 \dots, a_p \\ b_1, b_2 \dots, b_q \end{matrix} \middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\,(a_2)_k\,\dots(a_p)_k}{(b_1)_k\,(b_2)_k\,\dots(b_q)_k}\,\frac{x^k}{k!},$$

where $a_1,\, a_2, \dots, a_p,\, b_1,\, b_2, \dots, b_q$, are complex or real parameters, with $b_i \neq 0$, for all $1 \le i \le q$.
The following lemma is needed in the sequel.

Lemma 1. *Let m and j be two positive integers, and let* $d_{j,m} = {}_2F_1\left(\begin{matrix} -m, j-m \\ j-2m+2 \end{matrix} \middle|\, \frac{-1}{4}\right)$. *The following recurrence relation is satisfied by* $d_{j,m}$*:*

$$\begin{aligned} &4\binom{j-m-1}{j-2m}(j-2)\,(j-m)\,d_{j-2,m-1} + \binom{j-m}{j-2m+1}(j-1)\times \\ &(j-m-1)\,d_{j-1,m-1} + 4\binom{j-m-1}{j-2m-1}(j-1)\,(j-m)\,d_{j-1,m} \\ &-4\binom{j-m}{j-2m}\,j\,(j-m-1)\,d_{j,m} = 0. \end{aligned} \tag{9}$$

Proof. First, we set

$$e_{j,m} = j\binom{j-m}{j-2m} d_{j,m},$$

Hence, it suffices to prove that

$$(j-m)\, e_{j-2,m-1} + \frac{1}{4}(j-m-1)\, e_{j-1,m-1} + (j-m)\, e_{j-1,m} = (j-m-1)\, e_{j,m}. \tag{10}$$

The definition of the hypergeometric function ${}_2F_1$, enables one to write $e_{j,m}$ as

$$e_{j,m} = j\binom{j-m}{j-2m}\sum_{k=0}^{m}\frac{(-m)_k\,(j-m)_k\,(-\frac{1}{4})^k}{(2+j-2m)_k\ k!}.$$

Now:

$$\begin{aligned}
&(j-m)\, e_{j-2,m-1} + \frac{1}{4}(j-m-1)\, e_{j-1,m-1} + (j-m)\, e_{j-1,m} = \\
&(j-2)(j-m)\binom{-1+j-m}{j-2m}\sum_{k=0}^{m-1}\frac{(1-m)_k\,(-1+j-m)_k\,(-\frac{1}{4})^k}{(2+j-2m)_k\ k!} \\
&+\frac{1}{4}(j-1)(j-m-1)\binom{j-m}{1+j-2m}\sum_{k=0}^{m-1}\frac{(1-m)_k\,(j-m)_k\,(-\frac{1}{4})^k}{(3+j-2m)_k\ k!} \\
&+(j-1)(j-m)\binom{-1+j-m}{j-2m-1}\sum_{k=0}^{m}\frac{(-1+j-m)_k\,(-m)_k\,(-\frac{1}{4})^k}{(1+j-2m)_k\ k!}.
\end{aligned} \tag{11}$$

Relation (11) after performing some manipulations is turned into

$$\begin{aligned}
&(j-m)\, e_{j-2,m-1} + \frac{1}{4}(j-m-1)\, e_{j-1,m-1} + (j-m)\, e_{j-1,m} \\
&= j\,(j-m-1)\binom{j-m}{j-2m}\sum_{k=0}^{m}\frac{(j-m)_k\,(-m)_k\,(-\frac{1}{4})^k}{(2+j-2m)_k\ k!} \\
&= (j-m-1)\, e_{j,m}.
\end{aligned}$$

Lemma 1 is now proved. □

3. Connection formulae between Lucas polynomials and first and second kinds of Chebyshev polynomials

This section deals with developing two new connection formulae between Lucas polynomials and first and second kinds of Chebyshev polynomials. In fact, we show that these connection formulae involve hypergeometric functions of the type ${}_2F_1$ of certain arguments. These connection formulae are explicitly given in the following two theorems.

Theorem 1. *The following connection formula is valid:*

$$T_j(x) = 2^{j-1}\sum_{r=0}^{\lfloor \frac{j}{2}\rfloor}\delta_{j-2r}\ (-1)^r\binom{j}{r}\,{}_2F_1\left(\begin{matrix}-r, r-j\\ 1-j\end{matrix}\,\middle|\,\frac{-1}{4}\right)\ L_{j-2r}(x), \quad j\geq 0, \tag{12}$$

where

$$\delta_j = \begin{cases}\frac{1}{2} & j=0,\\ 1, & j>0.\end{cases}$$

Theorem 2. *The following connection formula is valid:*

$$U_j(x) = 2^j \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \delta_{j-2r}\ (-1)^r \binom{j}{r}\ {}_2F_1 \left(\begin{matrix} -r, r-j \\ -j \end{matrix} \,\middle|\, \frac{-1}{4} \right)\ L_{j-2r}(x), j \geq 0. \tag{13}$$

Proof. Due to the similarity of the proofs of Theorems 1 and 2, it is sufficient to prove Theorem 1.
Since

$${}_2F_1 \left(\begin{matrix} -r, r-j \\ 1-j \end{matrix} \,\middle|\, -\frac{1}{4} \right) = \sum_{k=0}^{r} \frac{(-r)_k\,(r-j)_k\,(-\frac{1}{4})^k}{(1-j)_k\,k!},$$

so to prove Theorem 1, we have to prove the following formula

$$T_j(x) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{r} \frac{\delta_{j-2r}\,(-1)^{k+r}\,2^{-1+j-2k}\binom{j}{r}(-r)_k(r-j)_k}{(1-j)_k\,k!} L_{j-2r}(x). \tag{14}$$

Let us assume the relation

$$\phi_j(x) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{r} \frac{\delta_{j-2r}\,(-1)^{k+r}\,2^{-1+j-2k}\binom{j}{r}(-r)_k(r-j)_k}{(1-j)_k\,k!} L_{j-2r}(x), \tag{15}$$

It is not difficult to see that $\phi_1(x) = x$, $\phi_2(x) = 2\,x^2 - 1$. We aim to prove that $\phi_j(x)$ satisfies the same recurrence relation of $T_j(x)$, for all $j \geq 1$.
Now, multiply both sides of (15) by $(2\,x)$ to get

$$2x\,\phi_j(x) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{r} \frac{\delta_{j-2r}\,(-1)^{k+r}\,2^{j-2k}\binom{j}{r}(-r)_k(r-j)_k}{(1-j)_k\,k!}\ x\,L_{j-2r}(x). \tag{16}$$

If we substitute by the recurrence relation

$$x\,L_j(x) = L_{j+1}(x) - L_{j-1}(x),$$

into (16), then we get

$$\begin{aligned} 2x\,\phi_j(x) = &\sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{r} \frac{\delta_{j-2r}\,(-1)^{k+r}\,2^{j-2k}\binom{j}{r}(-r)_k(r-j)_k}{(1-j)_k\,k!} L_{j-2r+1}(x) \\ &+ \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{r} \frac{\delta_{j-2r}(-1)^{k+r+1}\,2^{j-2k}\binom{j}{r}(-r)_k(r-j)_k}{(1-j)_k\,k!} L_{j-2r-1}(x). \end{aligned} \tag{17}$$

After performing some manipulations, relation (17) can be put in the from

$$\begin{aligned} 2x\,\phi_j(x) &= \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} \sum_{k=0}^{r} \frac{\delta_{j-2r-1}(-1)^{k+r}\,2^{j-2-2k}\,\binom{j-1}{r}(-r)_k(r-j+1)_k}{(2-j)_k\,k!} L_{j-2r-1}(x) \\ &+ \sum_{m=0}^{\lfloor \frac{j+1}{2} \rfloor} \sum_{k=0}^{m} \frac{\delta_{j-2r+1}\,(-1)^{k+r}\,2^{j-2k}\,\binom{j+1}{r}(-r)_k(r-j-1)_k}{(-j)_k\,k!} L_{j-2r+1}(x) \\ &= \phi_{j-1}(x) + \phi_{j+1}(x). \end{aligned}$$

But, since the Chebyshev polynomials $T_j(x)$, $j \geq 1$, are uniquely constructed by means of the recurrence relation:

$$2\,x\,T_j(x) = T_{j-1}(x) + T_{j+1}(x),\, T_1 = x,\, T_2(x) = 2\,x^2 - 1,$$

then, we should have $\phi_j(x) \equiv\ T_j(x),\ \forall\ j \geq 1$.
Theorem 1 is now proved. □

4. Lucas-first and second kinds Chebyshev connection formulae

In this section, we give the connection formulae between Lucas polynomials and Chebyshev polynomials of the first and second kinds. These formulae are the inversion formulae to those given in Section 3. Again, we will show that these formulae are also expressed in terms of hypergeometric functions of the type ${}_2F_1$. These results are given explicitly in the following two theorems.

Theorem 3. *The following connection formula is valid:*

$$L_j(x) = \frac{1}{2^{j-1}} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \delta_{j-2r} \binom{j}{r}\, {}_2F_1\left(\begin{matrix} -r, r-j \\ 1-j \end{matrix} \middle| -4 \right) T_{j-2r}(x), \quad j \geq 0. \tag{18}$$

Theorem 4. *The following connection formula is valid:*

$$L_j(x) = j \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j-r}{j-2r} \frac{2^{2r-j}}{j-r}\, {}_2F_1\left(\begin{matrix} -r, j-r \\ j-2r+2 \end{matrix} \middle| \frac{-1}{4} \right) U_{j-2r}(x), \quad j \geq 0. \tag{19}$$

Proof. We will prove Theorem 4 by induction. Assume that relation (19) is valid for $(j-2)$ and $(j-1)$, and we will show its validity for j. Starting with the recurrence relation of Lucas polynomials

$$L_j(x) = x\,L_{j-1}(x) + L_{j-2}(x),$$

then the application of the induction hypothesis twice on $L_{j-1}(x)$ and $L_{j-2}(x)$, and making use of the relation

$$x\,U_j(x) = \frac{1}{2}\left[U_{j-1}(x) + U_{j+1}(x)\right],$$

yield

$$\begin{aligned} L_j(x) =&(j-1) \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j-r-1}{j-2r-1} \frac{2^{2r-j}}{j-r-1}\, {}_2F_1\left(\begin{matrix} -r, j-r-1 \\ j-2r+1 \end{matrix} \middle| \frac{-1}{4} \right) \times \\ &\big(U_{j-2r-2}(x) + U_{j-2r}(x)\big) + (j-2) \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor -1} \binom{j-r-2}{j-2r-2} \frac{2^{-j+2r+2}}{j-r-2} \times \\ &{}_2F_1\left(\begin{matrix} -r, j-r-2 \\ j-2r \end{matrix} \middle| -\frac{1}{4} \right) U_{j-2r-2}(x). \end{aligned} \tag{20}$$

Relation (20), after performing some manipulations, takes the form

$$L_j(x) = \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} g_{j,r} U_{j-2r}(x) + \frac{1}{2}\, g_{j-1,\mu_j}\, U_{j-2\,\mu_j-2} + g_{j-2,\nu_j-1}\, U_{j-2\nu_j}\, \theta_j, \tag{21}$$

where

$$g_{j,r}=\frac{1}{2^{j+2-2r}}\Bigg\{\frac{4\binom{j-r-1}{j-2r}(j-2)}{j-r-1}\,{}_2F_1\left(\begin{matrix}1-r,j-r-1\\ j-2r+2\end{matrix}\,\middle|\,\frac{-1}{4}\right)+\frac{\binom{j-r}{j-2r+1}\,(j-1)}{j-r}\,{}_2F_1\left(\begin{matrix}1-r,j-r\\ j-2r+3\end{matrix}\,\middle|\,-\frac{1}{4}\right)+\frac{4\binom{j-r-1}{j-2r-1}\,(j-1)}{j-r-1}\,{}_2F_1\left(\begin{matrix}-r,j-r-1\\ j-2r+1\end{matrix}\,\middle|\,\frac{-1}{4}\right)\Bigg\},$$

and

$$\mu_j=\left\lfloor\frac{j-1}{2}\right\rfloor,\ \nu_j=\left\lfloor\frac{j}{2}\right\rfloor,\ \theta_j=\begin{cases}1, & \text{j even},\\ 0, & \text{j odd}.\end{cases}$$

Now, by means of of Lemma 1, the coefficients $g_{j,r}$ can be simplified to give

$$g_{j,r}=j\binom{j-r}{j-2r}\frac{2^{2r-j}}{j-r}\,{}_2F_1\left(\begin{matrix}-r,j-r\\ j-2r+2\end{matrix}\,\middle|\,\frac{-1}{4}\right).$$

Finally, after some manipulations, it can be shown that

$$L_j(x)=j\sum_{r=0}^{\lfloor\frac{j}{2}\rfloor}\binom{j-r}{j-2r}\frac{2^{2r-j}}{j-r}\,{}_2F_1\left(\begin{matrix}-r,j-r\\ j-2r+2\end{matrix}\,\middle|\,\frac{-1}{4}\right)U_{j-2r}(x).$$

□

5. Applications to the connection formulae between Lucas and Chebyshev polynomials and their inversion formulae

This section is dedicated to presenting three applications based on the results given in Sections 3 and 4. The first application exhibits new expressions involving Lucas numbers. The second exhibits some other new expressions for the derivatives sequences of Lucas numbers. In the third application, some weighted definite integrals involving certain products of Lucas and Chebyshev polynomials are evaluated.

5.1. **Some formulae involving Lucas numbers.** Some new formulae involving Lucas numbers are displayed. These expressions are direct consequences of the results in Sections 3 and 4.

Corollary 1. *The following two identities are valid:*

$$2^{j-1}\sum_{r=0}^{\lfloor\frac{j}{2}\rfloor}\delta_{j-2r}\,(-1)^r\binom{j}{r}\,{}_2F_1\left(\begin{matrix}-r,r-j\\ 1-j\end{matrix}\,\middle|\,\frac{-1}{4}\right)L_{j-2r}=1,\quad j\ge 0, \tag{22}$$

and

$$2^{j}\sum_{r=0}^{\lfloor\frac{j}{2}\rfloor}\delta_{j-2r}\,(-1)^r\binom{j}{r}\,{}_2F_1\left(\begin{matrix}-r,r-j\\ -j\end{matrix}\,\middle|\,\frac{-1}{4}\right)L_{j-2r}=j+1,\quad j\ge 0. \tag{23}$$

Corollary 2. *The following two expressions for Lucas numbers hold*

$$L_j=\frac{1}{2^{j-1}}\sum_{r=0}^{\lfloor\frac{j}{2}\rfloor}\delta_{j-2r}\binom{j}{r}\,{}_2F_1\left(\begin{matrix}-r,r-j\\ 1-j\end{matrix}\,\middle|\,-4\right),\quad j\ge 0, \tag{24}$$

and

$$L_j = j \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j-r}{j-2r} 2^{2r-j} \frac{(j-2r+1)}{(j-r)} \, {}_2F_1\left(\begin{matrix} -r, j-r \\ j-2r+2 \end{matrix} \middle| \frac{-1}{4} \right), \quad j \geq 0. \tag{25}$$

Proof. The proof of Corollaries 1 and 2, can be obtained immediately from formulae (12), (13), (18) and (19), respectively, by setting $x = 1$. □

Corollary 3. *For $i^2 = -1$, and for every nonnegative integer j, the following identities hold:*

$$i^j L_j = 2^j \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^r \binom{j}{r} \delta_{j-2r} \, {}_2F_1\left(\begin{matrix} -r, -j+r \\ 1-j \end{matrix} \middle| \frac{-1}{4} \right) L_{j-2r}\left(\frac{i}{2}\right), \tag{26}$$

$$(-i)^j L_{3j} = 2^j \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^r \binom{j}{r} \delta_{j-2r} \, {}_2F_1\left(\begin{matrix} -r, -j+r \\ 1-j \end{matrix} \middle| \frac{-1}{4} \right) L_{j-2r}(-2i), \tag{27}$$

$$i^j F_{j+1} = 2^j \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^r \binom{j}{r} \delta_{j-2r} \, {}_2F_1\left(\begin{matrix} -r, -j+r \\ -j \end{matrix} \middle| \frac{-1}{4} \right) L_{j-2r}\left(\frac{i}{2}\right), \tag{28}$$

$$(-i)^j F_{3j+3} = 2^{j+1} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^r \binom{j}{r} \delta_{j-2r} \, {}_2F_1\left(\begin{matrix} -r, -j+r \\ -j \end{matrix} \middle| \frac{-1}{4} \right) L_{j-2r}(-2i). \tag{29}$$

Proof. Identities (26)-(29) follow immediately form the connection formulae (12), (13), (18) and (19), along with identities (5)-(8). □

5.2. **New derivatives sequences formulae.** Based on the connection formulae, (12), (13), (18) and (19), along with some properties of Chebyshev polynomials, some formulae of the derivatives sequences of Lucas numbers can be deduced. These formulae are given explicitly in the following two corollaries.

Corollary 4. *If $q \geq 1$, then the following two formulae hold:*

$$\sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^r \binom{j}{r} \delta_{j-2r} \, {}_2F_1\left(\begin{matrix} -r, r-j \\ 1-j \end{matrix} \middle| \frac{-1}{4} \right) L_{j-2r}^{(q)} = \frac{(-1)^{q+1} \sqrt{\pi}\, j^2 \, (1-j)_{q-1} \, (j+1)_{q-1}}{2^{q+j-1} \, \Gamma\left(q+\frac{1}{2}\right)}, \tag{30}$$

and

$$\sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^r \binom{j}{r} \, {}_2F_1\left(\begin{matrix} -r, -j+r \\ -j \end{matrix} \middle| \frac{-1}{4} \right) L_{j-2r}^{(q)} = \frac{(-1)^{q+1} \sqrt{\pi} \, (j)_3 \, (1-j)_{q-1} \, (j+3)_{q-1}}{2^{j+q+1} \, \Gamma\left(q+\frac{3}{2}\right)}. \tag{31}$$

Corollary 5. *If $q \geq 1$, then the following two formulae hold:*

$$
L_j^{(q)} = \frac{(-1)^{q+1}\sqrt{\pi}}{2^{q+j-1}\,\Gamma\left(q+\frac{1}{2}\right)} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \delta_{j-2r} \binom{j}{r} (j-2r)^2\, (j-2r+1)_{q-1} \times (-j+2r+1)_{q-1} \times {}_2F_1\left(\left. \begin{matrix} -r, r-j \\ 1-j \end{matrix} \right| \frac{-1}{4} \right), \tag{32}
$$

and

$$
L_j^{(q)} = \frac{\sqrt{\pi}\,(-1)^{q+1}\, j}{2^{q+1}\,\Gamma\left(q+\frac{3}{2}\right)} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j-r}{j-2r} \frac{2^{2r-j}}{j-r} (j-2r)_3\, (j-2r+3)_{q-1} \times (-j+2r+1)_{q-1}\, {}_2F_1\left(\left. \begin{matrix} -r, j-r \\ j-2r+2 \end{matrix} \right| \frac{-1}{4} \right). \tag{33}
$$

Proof. The proof of formulae (30)-(33), can be followed, if we differentiate the connection formulae (12), (13), (18) and (19) with respect to x, setting $x = 1$, and making use of the identities (3) and (4). □

5.3. **Some integrals formulae involving Chebyshev and Lucas polynomials.** The following two integrals formulae can be obtained with the aid of the Theorems 3 and 4.

Corollary 6. *For all $j \geq k$, the following integrals formulae are valid:*

$$
\int_{-1}^{1} \frac{L_j(x) T_k(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} \frac{\pi \binom{j}{\frac{j-k}{2}}}{2^j\, c_k}\, {}_2F_1\left(\left. \begin{matrix} \frac{-(j+k)}{2}, \frac{k-j}{2} \\ 1-j \end{matrix} \right| -4 \right), & (j+k) \text{ even}, \\ 0, & \text{otherwise}, \end{cases} \tag{34}
$$

and

(35)

$$
\int_{-1}^{1} \sqrt{1-x^2}\, L_j(x)\, U_k(x)\, dx = \begin{cases} \frac{\pi\, j \binom{\frac{j+k}{2}}{k}}{2^k\, (j+k)}\, {}_2F_1\left(\left. \begin{matrix} \frac{k-j}{2}, \frac{j+k}{2} \\ k+2 \end{matrix} \right| \frac{-1}{4} \right), & (j+k) \text{ even}, \\ 0, & \text{otherwise}. \end{cases}
$$

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